First-price auction with three bidders

Consider a first-price auction with three bidders. Their valuations are ordered as

$$v_1 > v_2 > v_3$$

Assume that, in the event of a tie, the object is assigned to the bidder with the highest valuation.

1. Indicate whether the bid profile $(b_1, b_2, b_3) = (v_1, v_1, 0)$ constitutes a Nash equilibrium
2. Explain rigorously why bidding more than one's own valuation is weakly dominated by bidding one's own valuation. You may assume arbitrary numerical values for the valuations, but you should state them explicitly
3. Show that bidding one's own valuation is weakly dominated by any lower bid
4. Find five Nash equilibria. Indicate which of them seems the most reasonable

Solution

1. Yes, the bid profile

$$(b_1, b_2, b_3) = (v_1, v_1, 0)$$

is a Nash equilibrium

At this profile, bidders 1 and 2 tie at the highest bid v_1. Since ties are broken in favor of the bidder with the highest valuation, bidder 1 wins the object

Because this is a first-price auction, the winner pays her own bid. Therefore, the payoffs are

$$u_1 = v_1 - v_1 = 0$$

$$u_2 = 0$$

$$u_3 = 0$$

We now check that no bidder has a profitable unilateral deviation

Bidder 1

If bidder 1 deviates to some bid $b_1' < v_1$, then bidder 2 becomes the unique highest bidder with bid v_1, so bidder 1 loses and gets payoff

$$0$$

If bidder 1 deviates to some bid $b_1' > v_1$, she still wins, but now her payoff is

$$v_1 - b_1' < 0$$

So bidder 1 cannot improve upon her current payoff 0

Bidder 2

If bidder 2 deviates to some bid $b_2' < v_1$, she loses, so her payoff remains

$$0$$

If bidder 2 deviates to some bid $b_2' > v_1$, she wins, but then her payoff is

$$v_2 - b_2'$$

Since $v_2 < v_1 < b_2'$, this payoff is negative

Thus bidder 2 also has no profitable deviation

Bidder 3

If bidder 3 deviates to some bid $b_3' \leq v_1$, she still loses, because either bidder 1 wins with bid v_1, or bidder 1 wins the tie at v_1

So her payoff remains

$$0$$

If bidder 3 deviates to some bid $b_3' > v_1$, she wins, but then her payoff is

$$v_3 - b_3'$$

Since $v_3 < v_1 < b_3'$, this payoff is negative

Therefore bidder 3 has no profitable deviation either

Since no bidder can improve her payoff by deviating unilaterally, we conclude that

$$(b_1, b_2, b_3) = (v_1, v_1, 0) \text{ is a Nash equilibrium}$$

2. Fix any bidder i, with valuation v_i, and consider an arbitrary bid

$$\hat{b}_i > v_i$$

We will show that bidding v_i weakly dominates bidding $\hat{b}_i$

Let

$$m_{-i} = \max\{b_j : j \neq i\}$$

denote the highest bid among the other bidders

In a first-price auction, if bidder i wins with bid b_i, her payoff is

$$u_i = v_i - b_i$$

If she loses, her payoff is

$$u_i = 0$$

We now compare the payoff from bidding v_i with the payoff from bidding $\hat{b}_i > v_i$, for every possible value of m_{-i}

Case 1: $m_{-i} < v_i$

If bidder i bids v_i, she wins and obtains

$$u_i(v_i, b_{-i}) = v_i - v_i = 0$$

If instead she bids $\hat{b}_i$, she also wins, but her payoff is

$$u_i(\hat{b}_i, b_{-i}) = v_i - \hat{b}_i < 0$$

Hence,

$$u_i(v_i, b_{-i}) > u_i(\hat{b}_i, b_{-i})$$

Case 2: $m_{-i} = v_i$

If bidder i bids v_i, she either loses the tie and gets 0, or wins the tie and pays v_i, which also gives payoff 0

Thus,

$$u_i(v_i, b_{-i}) = 0$$

If instead she bids $\hat{b}_i > v_i$, she becomes the unique highest bidder and wins, so her payoff is

$$u_i(\hat{b}_i, b_{-i}) = v_i - \hat{b}_i < 0$$

Therefore,

$$u_i(v_i, b_{-i}) > u_i(\hat{b}_i, b_{-i})$$

Case 3: $v_i < m_{-i} < \hat{b}_i$
If bidder i bids v_i, she loses and obtains

$$u_i(v_i, b_{-i}) = 0$$

If instead she bids $\hat{b}_i$, she wins and pays $\hat{b}_i$, so

$$u_i(\hat{b}_i, b_{-i}) = v_i - \hat{b}_i < 0$$

Hence,

$$u_i(v_i, b_{-i}) > u_i(\hat{b}_i, b_{-i})$$

Case 4: $m_{-i} = \hat{b}_i$
If bidder i bids v_i, she loses and gets

$$u_i(v_i, b_{-i}) = 0$$

If she bids $\hat{b}_i$, there is a tie at $\hat{b}_i$. She may lose the tie, in which case her payoff is 0, or win the tie, in which case her payoff is

$$v_i - \hat{b}_i < 0$$

Thus,

$$u_i(v_i, b_{-i}) \geq u_i(\hat{b}_i, b_{-i})$$

Case 5: $m_{-i} > \hat{b}_i$
In this case, bidder i loses under both bids, so

$$u_i(v_i, b_{-i}) = u_i(\hat{b}_i, b_{-i}) = 0$$

We have shown that for every possible profile of the opponents' bids,

$$u_i(v_i, b_{-i}) \geq u_i(\hat{b}_i, b_{-i})$$

and in Cases 1, 2, and 3 the inequality is strict

Therefore, bidding v_i weakly dominates any bid $\hat{b}_i > v_i$

Hence, in a first-price auction, any bid above one's own valuation is weakly dominated by bidding one's own valuation

3. Fix any bidder i, with valuation v_i, and consider an arbitrary bid

$$\hat{b}_i > v_i$$

We will show that bidding v_i weakly dominates bidding $\hat{b}_i$

Let

$$m_{-i} = \max\{b_j : j \neq i\}$$

denote the highest bid among the other bidders

In a first-price auction, if bidder i wins with bid b_i, her payoff is

$$u_i = v_i - b_i$$

If she loses, her payoff is

$$u_i = 0$$

We now compare the payoff from bidding v_i with the payoff from bidding $\hat{b}_i > v_i$, for every possible value of m_{-i}

Case 1: $m_{-i} < v_i$

If bidder i bids v_i, she wins and obtains

$$u_i(v_i, b_{-i}) = v_i - v_i = 0$$

If instead she bids $\hat{b}_i$, she also wins, but her payoff is

$$u_i(\hat{b}_i, b_{-i}) = v_i - \hat{b}_i < 0$$

Hence,

$$u_i(v_i, b_{-i}) > u_i(\hat{b}_i, b_{-i})$$

Case 2: $m_{-i} = v_i$

If bidder i bids v_i, she either loses the tie and gets 0, or wins the tie and pays v_i, which also gives payoff 0

Thus,

$$u_i(v_i, b_{-i}) = 0$$

If instead she bids $\hat{b}_i > v_i$, she becomes the unique highest bidder and wins, so her payoff is

$$u_i(\hat{b}_i, b_{-i}) = v_i - \hat{b}_i < 0$$

Therefore,

$$u_i(v_i, b_{-i}) > u_i(\hat{b}_i, b_{-i})$$

Case 3: $v_i < m_{-i} < \hat{b}_i$

If bidder i bids v_i, she loses and obtains

$$u_i(v_i, b_{-i}) = 0$$

If instead she bids $\hat{b}_i$, she wins and pays $\hat{b}_i$, so

$$u_i(\hat{b}_i, b_{-i}) = v_i - \hat{b}_i < 0$$

Hence,

$$u_i(v_i, b_{-i}) > u_i(\hat{b}_i, b_{-i})$$

Case 4: $m_{-i} = \hat{b}_i$
If bidder i bids v_i, she loses and gets

$$u_i(v_i, b_{-i}) = 0$$

If she bids $\hat{b}_i$, there is a tie at $\hat{b}_i$. She may lose the tie, in which case her payoff is 0, or win the tie, in which case her payoff is

$$v_i - \hat{b}_i < 0$$

Thus,

$$u_i(v_i, b_{-i}) \geq u_i(\hat{b}_i, b_{-i})$$

Case 5: $m_{-i} > \hat{b}_i$
In this case, bidder i loses under both bids, so

$$u_i(v_i, b_{-i}) = u_i(\hat{b}_i, b_{-i}) = 0$$

We have shown that for every possible profile of the opponents' bids,

$$u_i(v_i, b_{-i}) \geq u_i(\hat{b}_i, b_{-i})$$

and in Cases 1, 2, and 3 the inequality is strict

Therefore, bidding v_i weakly dominates any bid $\hat{b}_i > v_i$

Hence, in a first-price auction, any bid above one's own valuation is weakly dominated by bidding one's own valuation

4. From part (1), we know that in a first-price auction with three bidders and tie-breaking in favor of the highest valuation, every pure-strategy Nash equilibrium must have the following form:

$$b_1 = p \qquad \max\{b_2, b_3\} = p \qquad v_2 \leq p \leq v_1$$

with bidder 1 winning by the tie-breaking rule

Therefore, many equilibria exist. For example, the following five bid profiles are Nash equilibria:

$$(b_1, b_2, b_3) = (v_2, v_2, 0)$$

$$(b_1, b_2, b_3) = (v_2, v_2, v_3)$$

$$(b_1, b_2, b_3) = \left(\frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2}, 0\right)$$

$$(b_1, b_2, b_3) = (v_1, v_1, 0)$$

$$(b_1, b_2, b_3) = (v_2, 0, v_2)$$

Each of these profiles fits the general equilibrium characterization:

- bidder 1 bids the common highest bid p
- at least one of the other bidders also bids p
- the common highest bid satisfies $v_2 \leq p \leq v_1$
- bidder 1 receives the object because ties are broken in favor of the highest valuation

Hence, each of the five profiles above is a Nash equilibrium

Now we ask which of them seems the most reasonable

Taking into account parts (2) and (3), bids above one's own valuation are weakly dominated, and bidding exactly one's own valuation is itself weakly dominated by any slightly lower bid

This means that no pure Nash equilibrium is completely free from the use of weakly dominated strategies

Still, among all equilibria, the most reasonable ones are those with the smallest possible highest bid, namely

$$p = v_2$$

The reason is that these equilibria avoid the most implausible behavior, namely bidders submitting bids strictly above their own valuations

Among them, the simplest and most natural profile is

$$(b_1, b_2, b_3) = (v_2, v_2, 0)$$

In this equilibrium, bidder 1 wins, the winning bid is just enough to deter bidder 2, and bidder 3 stays out by bidding 0

Thus, although even this profile involves weakly dominated behavior by bidder 2, it is still the least unreasonable and most focal among the pure Nash equilibria

Therefore, a natural answer is that $(v_2, v_2, 0)$ is the most reasonable among the five equilibria